COVERING ANNULI BY PLANKS

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1. Description of the problem

Plank problems form a classical topic in convex geometry. A *plank* (or strip) in the plane is the region between two parallel lines, whose distance apart is called the *width* of the plank. The original plank problem of Tarski, which dates back roughly a century, reads as follows.

Question 1. *Assume that a circular disc D is covered by a finite collection of planks. Prove that the sum of the widths of these planks is at least the diameter of D.*

Although this question has a very elegant solution, its generalization for arbitrary planar convex sets is not so easy to prove, and that was achieved only by Bang in the 1950's. He in fact proved the result for arbitrary convex bodies in $\mathbb{R}^d$, and at the same time, found a natural generalization of the problem which, in full generality, have been open ever since.

In the research project, we will concentrate on a variant of the question that was suggested by A. Bezdek. What happens, if one doesn't need to cover the entire disc, but only an outer annulus of it? Let D now denote the unit disc centered at the origin, i.e. the disc of radius 1 with center $(0, 0)$.

Question 2. *Let $0 < \varepsilon < 1$ be a given parameter. Is it true that if ε is sufficiently small, then if a finite collection of planks cover the annulus $D \setminus \varepsilon D$, then the sum of widths of the planks is still at least 2?*

Bezdek proved the statement for a square instead of the disc. Later, White and Wisewell extended the result for all polygons. However, Question 2 is open for any convex, non-polygonal planar set.

The main goal of the research project is to answer Question 2, and to study further questions related to plank problems.

2. Prerequisites

Although the problem sounds very elementary, the applied tools are based on various methods from geometry, analysis and combinatorics. That said, we are going to start from the basics and cover topics that are needed. Completion of an introductory calculus course is mandatory; a course in elementary geometry is preferred.

3. Qualifying problems

In order to participate in the research project, you will need to submit solutions for the set of qualifying problems below by no later than August 27, to the email address `ambruge AT gmail DOT com`.

Qualifying Problem 1. *Find an $\varepsilon < 1$ for which the statement of Question 2 does not hold. Try to make ε as small as possible.*

Qualifying Problem 2. *Let $n \geq 2$ be an integer, and $u_1, \ldots, u_n \in \mathbb{R}^2$ be unit vectors, i.e. vectors of unit length in the plane. Assume that for each $i \in [n]$, the plank P_i is of the form*

$$P_i = \{x \in \mathbb{R}^2 : |\langle x, u_i \rangle| \leq 1\}.$$

That is, P_i is orthogonal to u_i, it has width 2, and it is centrally symmetric with respect to the origin. Prove that there exists a choice of signs $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$ such that the point

$$u = \varepsilon_1 u_1 + \ldots + \varepsilon_n u_n$$

is not covered by the interior of any of the planks P_i, that is,

$$|\langle u, u_i \rangle| \geq 1$$

*holds for each i. (*Hint: try to select the signs for which the norm of u is maximal.*) If you do not manage to prove the statement, try to show it for $n = 2, 3$ by hand.*

Qualifying Problem 3. *Try to extend Question 2 for triangles as follows. First, let T be a triangle. with minimal altitude t. You may assume that if a set of planks cover T, then the sum of their widths is at least t. Now, try to find a triangle T with the following property: for any positive $0 < \varepsilon < 1$ one may delete from T a small homothetic copy of T strictly contained within T so that the remaining set can be covered by a set of planks the sum of whose widths is strictly less than t. Try to characterize triangles with this property.*

Qualifying Problem 4. *(***extra problem***) Prove the original plank problem, Question 1, by taking a ball in $\mathbb{R}^3$ whose equator is the boundary of D, then projecting up the planks onto the boundary of the ball, and calculating the amount of surface area covered by such a spherical plank.*