Graph connectivity: orientations, removable subgraphs, algorithms

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Introduction

An important graph parameter is the edge- (or vertex-) connectivity of a graph. We say that a graph G is *k-edge-connected* (*k-vertex-connected*) if it is not possible to disconnect G by the deletion of less than k edges (vertices, resp.).

Menger's fundamental theorem says that $G = (V, E)$ is k-edge-connected (k-vertex connected) if and only if there exist k pairwise edge-disjoint (internally vertex-disjoint, resp.) paths between vertices u, v, for all pairs $u, v \in V$. Similar statements hold for directed graphs. There exist efficient algorithms for testing whether a graph is k-edge or vertex connected, for example by using network flows.

In this project we shall study variations, potential extensions, and the algorithmic aspects of the following recent results on graph connectivity. The *orientation* of a graph G is a directed graph D obtained from G by replacing every edge uv by an oriented edge (or *arc*) with tail u and head v, or with tail v and head u. A rich research area is concerned with the characterization of graphs that have orientations D satisfying specific conditions on the connectivity of D or the degrees of the vertices in D, or both. For example, a classic result of Robbins says that G has a strongly connected orientation if and only if G is 2-edge-connected. (A digraph is *strongly connected*, if there is a directed path from u to v, for all ordered pairs u, v of vertices.) It is not hard to see that a strongly connected orientation, if it exists, can be found quickly by performing a DFS search on G. A more recent, and much deeper result of Thomassen characterizes graphs with a 2-vertex-connected orientation. The algorithmic aspects of the latter result are unexplored. Finding efficient algorithms would be interesting even in some natural special cases. The corresponding questions for higher connectivity are open. There are also several open questions concerning the rooted versions (where we require that the orientation contains k disjont paths from a designated root vertex to every other vertex) and the potential extensions to hypergaphs.

Another interesting set of problems is concerned with extra conditions, which guarantee that certain subgraphs (vertices, edges, paths, cycles, and so on) can be removed from a k-edge-connected graph G without destroying k-edge-connectivity. For example, a basic result of this kind, due to Mader, is that if the minimum degree of a k-edge-connected graph G is at least $k+1$, then there exists a removable edge. A deeper result claims that minimum degree at least $k+2$ gives rise to a removable cycle. There are several open problems in this area, including conjectures from the 70's. In some cases the existence a removable subgraph is known, but the algorithmic aspects are open (i.e. how to find such a removable subgraph in an efficient manner).

Methods and prerequisites

Familiarity with the basics of graph theory is useful.

Warm up exercises

Solve the next five exercises, and hand in the solutions by email, before the start of the research project. In the following exercises the graphs may contain parallel edges, but not loops.

Exercise 1. We are given a graph $G = (V, E)$ and a vertex $s \in V$. When does G have an orientation in which there is a directed path from s to every other vertex? Characterize the pairs G, s, for which such an orientation exists.

Exercise 2. Suppose that in a digraph $D = (V, A)$ there exist k pairwise internally vertex-disjoint paths from a designated root vertex r to every other vertex $v \in V - r$. Show that D has a spanning subdigraph $D' = (V, A')$ in which there exist k pairwise internally vertex-disjoint paths from r to every other vertex $v \in V - r$ and the number of arcs entering a vertex v is equal to k for each $v \in V - r$.

A vertex set S in a graph G is a *separator* if $G - S$ is disconnected. It is *minimal*, if no proper subset of S is a separator.

Exercise 3. A graph G is called *chordal* if every cycle C in G of length at least four has a chord (that is, an edge of G which connects two non-consecutive vertices of the cycle). Let S be a minimal separator in a chordal graph G. Prove that the vertices in S are pairwise adjacent.

Exercise 4. Prove that the edge set of a graph G can be partitioned into (the edge sets of) two spanning trees of G if and only if G can be obtained from a single vertex by applying the following operations:
(i) add a new vertex v to the graph and two new edges incident with v,
(ii) delete an edge xy from the graph, and add a new vertex v and three new edges vx, vy, vz incident with v (thus the three new edges must include the edges from v to the endvertices of the deleted edge).

Exercise 5. Characterize the connected graphs G for which there is an orientation D of G in which each vertex is the head of at most one arc (in other words, the *in-degree* of each vertex v is at most one).

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