1 Stick Knots and Colorings

Knot theory is an interesting and dynamically evolving field of topology. A *knot* is a circle embedded in the three-dimensional space, and the main question of this theory is how to distinguish knots and determine from their planar projections (*knot diagrams*) whether they represent equivalent knots. For this purpose, we use *invariants*, and one of the most basic such properties is p-colorability for an odd prime number p.

We say that a projection is *p-colorable* if we can label its strands using numbers $0, ..., p-1$, such that at least two of the labels are distinct and at each crossing the relation

$$2x - y - z \equiv 0 \pmod{p}$$

holds, where x is the label on the overcrossing and y and z the other two labels.

With algebraic methods, we can tell for which primes a given knot is p-colorable. However, in most p-colorings, there is no need to use all p colors.

Conjecture 1. *Let p be an odd prime, and K any p-colorable knot. The minimum number of colors needed in a p-coloring of K is $\lfloor \log_2 p \rfloor + 2$.*

There are several studies on this question, and the conjectured number is known to be a lower bound. After getting familiar with the neccessary background and tools, we plan to work on this problem.

A *stick knot* is a knot which consists of a finite union of line segments called *sticks*. A natural question is the minimal number of sticks needed to construct a representative of a given knot type, that is, the *stick number* $s(K)$ of a knot K. For most knots, we only know weak bounds on the stick number. Motivated by this, we aim to find the exact number or better bounds at least for special knot types such as *twist knots* or *two-bridge knots*.

Conjecture 2. *Denote the* crossing number *(the minimal number of double points in any generic planar projection of the knot type) by $c(K)$. For any two-bridge knot K with $c(K) \geq 6$,*

$$s(K) = c(K) + 2.$$

The participants will learn the basic notions and results of knot theory, along with some useful combinatoric and algebraic tools, as well as knot invariants. They get an insight to related, more specific fields of knot theory such as *grid diagrams*, *contact structures* or *Legendrian knots*.

2 Convergence of randomized distributed averaging

Some agents wish to average their measurements. If they could communicate synchronously along a nice graph, this task would proceed quickly. Instead, they are only allowed to send messages on few of the edges at a time. A functioning scheme for this scenario is the so-called ratio consensus (also known as push-sum or weighted gossip), where each entity i stores a secondary variable w^i alongside its value x^i. Here x_0^i is initialized as the measurement, $w_0^i = 1$. For a message from i to j, the update is as follows:

$$x_{t+1}^i = (1-q)x_t^i, \qquad x_{t+1}^j = x_t^j + qx_t^i,$$
$$w_{t+1}^i = (1-q)w_t^i, \qquad w_{t+1}^j = w_t^j + qw_t^i,$$

where $q \in (0,1)$ is a fixed global ratio. It has been proven that for nontrivial i.i.d. messaging, the ratio x_t^i/w_t^i tends to the average for each entity with probability 1, exponentially fast. But at what rate?

To get a situation where we can grasp this rate, we consider the case where each agent sends a message to a single random neighbor at each step. Let P be the matrix of transmission probabilities, λ_i its spectrum, and n the number of entities. In the transitive case (and perhaps more generally?), with some effort, we can obtain the largest root of the following (polynomial) equation as an upper bound for the rate:

$$\frac{q^2}{n}\sum_{j>1}\frac{1-\lambda_j^2}{x-(1-q+q\lambda_j)^2} = 1.$$

Whether this is elegant or obscure is up to taste, but certainly practical as it only requires understanding a single root of an $n-1$ degree polynomial. For other known approaches, one needs to understand an $n^2 \times n^2$ matrix or asymptotic quantities.

Here a challenge is to get a cleaner insight on this root or a robust approximation, which is less implicit but still informative (i.e., something away from 1), possibly some description of the dependence on q.

Background, references:

- Generally about the averaging process https://ieeexplore.ieee.org/iel5/8767/27770/01238221.pdf,
- on rate estimates https://ieeexplore.ieee.org/iel7/9/9743955/09382110.pdf, https://arxiv.org/pdf/2104.04802,
- and for the polynomial estimate mentioned above https://arxiv.org/pdf/2307.06157.

3 Pattern-free permutons

Using a natural subpermutation concept one can define the density of a short permutation or pattern in a long permutation, and thus two large permutations can be considered similar if their pattern densities are close. Following these ideas, a permutation sequence is called convergent if the sequence of pattern densities converges for any pattern. The picture is complete due to the fact that there is a nice limit object: the limits of convergent permutation sequences are certain probability measures defined on $[0, 1]^2$, which are called permutons. Sampling n points from such a measure also determines a pattern, through which the density of patterns can be defined in permutons. An interesting question is what can be said about permutations or permutons in which the density of one or more given patterns is fixed. To mention some already solved problems: how many permutations are there that avoid the pattern 123, or what could be the support of a 123-free permuton? The beauty of the theory lies in the fact that a connection can be established between such discrete-continuous pairs of questions. The aim of research is to better understand this relationship.

4 Size of certain symmetric differences

It is an open question in combinatorial geometry whether the area of the symmetric difference of an odd number of unit discs is always at least π or not. We would look at discrete analogues of similar flavour. For which classes of finite sets $A \subseteq \mathbb{Z}^r$ is it true that the size of an odd number of translates of A is always at least $|A|$? Also, for which classes can we give good lower bounds if we can take *any (finite) number* of translates of A? An interesting special case is known as Pilz' conjecture which states that the size of the symmetric difference of the sets $A, 2A, \ldots, nA$ is at least n if $A \subseteq \mathbb{N}$ is finite and $kA = \{ka : a \in A\}$. The best known lower bound is of shape $n/(\log n)^\lambda$ (where $\lambda \approx 0.2223$).

5 Additive combinatorics

We would study the largest possible size of sets avoiding certain arithmetic configurations in $\mathbb{F}_p^n$, or more generally, in $\mathbb{Z}_m^n$. A few example of forbidden configurations: k-term arithmetic progressions or corners: 3-element sets of the form $(a, b), (a + d, b), (a, b + d)$ in $\mathbb{F}_p^n \times \mathbb{F}_p^n$.

We briefly mention some very recent developments in the area. In $\mathbb{F}_3^n$ a lower bound of $2.2202^{n-o(n)}$ was recently obtained for sets avoiding 3-term arithmetic progressions by Romera– Paredes et al. [3] using artificial intelligence building upon traditional methods from previous bounds. In a parallel (yet unpublished) work, Naslund obtained the lower bound $2.2208^{n-o(n)}$ with an approach related to Shannon capacity. Elsholtz et al [1] proved that for any fixed integer $m \geq 2$ and sufficiently large n (in terms of m), there exists a three-term progression free subset $A \subseteq \mathbb{Z}_m^n$ of size $|A| \geq (cm)^n$ for some absolute constant $c > 1/2$. Building on their ideas Hunter [2] gave the first quasipolynomial improvement since the original construction of Behrend for the size of sets avoiding 3-term arithmetic progressions in $\{1, 2, \ldots, n\}$.

Our aim is to improve on the known bounds in certain settings.

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6 Counting maximal independent sets

Recall that a vertex set in a (hyper)graph is *independent* if it contains no edge. An independent set is *maximal* if it is not a proper subset of a larger independent set. Let MIS(G) denote the number of maximal independent sets (MIS) in a graph G. Miller and Muller [5] and independently Moon and Moser [6] showed that for all n-vertex graphs G we have $\mathrm{MIS}(G) \leq 3^{n/3}$ which is sharp as given by the vertex-disjoint union of triangles. When triangles are forbidden from G, Hujter and Tuza [3] showed $\mathrm{MIS}(G) \leq 2^{n/2}$ which is achievable by a matching. If we allow at most t vertex-disjoint triangles, then Palmer and Patkós [8] showed that the best is (roughly) to take t vertex-disjoint triangles and a matching on the remaining vertices.

There are several natural generalizations of these problems. The first is for 3-uniform hypergraphs (i.e. all hyperedges are of cardinality 3).

Problem 3. *Determine the maximum number of MIS in an n-vertex 3-uniform hypergraph. What if we require the hypergraph to be K_4^3-free?*

Disjoint copies of K_5^3 (construction by Tomescu [9]) gives a lower bound of about 1.5849^n and Lonc and Truszczyński [4] gave an upper bound of about 1.6702^n. Note that there are alternative definitions of an independent set in a hypergraph that allow further variations of the problem.

There are many generalizations in the graph setting. The first is to count induced r-regular subgraphs. An independent set is an induced 0-regular subgraph. The maximum number of maximimal induced 1-regular subgraphs (i.e. maximal induced matchings) in an n-vertex graph is known to be $10^{n/5}$ (see Basavaraju et al. [1]). This is achieved by disjoint copies of K_5. If we forbid triangles, the maximum is $3^{n/3}$ which is achieved by disjoint copies of $K_{3,3}$.

Problem 4. *What is the max number of induced matchings in a K_4-free n-vertex graph?*

Nielsen [7] showed that the maximum number of MIS of size k in an n-vertex graph is asymptotic to $(n/k)^k$. He, Nie and Spiro [2] examined the question when G is taken to be K_t-free. Among others they constructed an n-vertex triangle-free graph with $\Omega(n^{k/2})$ MIS of size $k \geq 4$ and asked for a matching upper-bound:

Problem 5. *Show that max number of MIS of size $k \geq 4$ in a triangle-free graph is $O(n^{k/2})$.*

One more related direction:

Problem 6. *Find the max number of maximal triangle-free sets in an n-vertex K_4-free graph.*

A good starting point for these problems is the beautiful inductive argument in [10].

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7 Euclidean Order-Ramsey Theory

Consider an ordered point set in your favorite metric space, e.g., $P = (p_1, \dots, p_k) \subset \mathbb{R}^d$. Is it true that for any ordering $\prec$ of the points of the metric space there is a copy of P where $p_1 \prec \ldots \prec p_k$? For example, it is true for $d = 2$ and $k = 3$ that for any ordering $\prec$

of the points of the plane we can find three points A, B, C that form the three vertices of a unit side-length equilateral triangle in this counterclockwise order, such that $A \prec B \prec C$. In general, we can be less restrictive, and only look for isometric copies of P, i.e., ignore the orientation and also allow reflections. Would that make the statement true for any triangle?

While this question has not been hitherto studied, there is a very rich theory of Euclidean Ramsey Theory (ERT), where instead of orderings we consider colorings. A good place to get acquainted with this topic is Graham's survey [6] and the three original papers by Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus [1, 2, 3]. The recordings of a more recent workshop on the topic can be found at https://coge.elte.hu/WERT.html.

The goal of this project is to combine ERT with the new field of the theory of ordered graphs [9]. In the main question of ERT is for which sets P it is true that for every r there is a d such that every r-coloring of $\mathbb{R}^d$ contains a monochromatic copy of P. It was conjectured in [1] that this is true exactly for spherical sets, though this has been later challenged [7]. Could something similar hold in case of orderings?

This might be more related to the Erdős-Rado Canonization Theorem [4] where the number of colors is not restricted. Similar questions have been considered in a geometric setting only very recently [8].

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8 Orientations minimizing the number of Eulerian tours

It is well-known that an oriented graph has an Eulerian tour (a walk that uses each arc exactly once) if and only if the orientation is Eulerian, that is, each vertex has equal in- and out-degrees. Suppose that we have an unoriented graph in which each degree is even. We would like to study the following question: Which Eulerian orientation of the graph has a minimal number of Eulerian tours? There is a conjecture for planar graphs. If a graph is embedded into the plane, then one can orient it in an Eulerian way so that around each vertex, in- and out-edges alternate. (Actually, there are two such orientations, that are "mirror images" of each other.) The conjecture is that these alternating orientations minimize the number of Eulerian tours. There is no conjecture about which orientations could minimize the number of Eulerian tours for non-planar graphs. The goal would be to look at many examples, and set up many conjectures (and ideally, prove some of them). For example: How many minimizing orientations are there for a typical graph? How do the minimizing orientations relate to each other? Can we say anything about the orientations maximizing the number of Eulerian tours?